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Stochastic Modeling of an Aircraft Traversing a Runway Using Time Series Analysis

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Time series, from a narrow point of view, is a sequence of observations on a stochastic process made at discrete and equally spaced time intervals. Its future behavior can be predicted by identifying, fitting, and confirming a mathematical model. In this paper, time series analysis is applied to problems concerning runway-induced vibrations of an aircraft. A simple mathematical model based on this technique is fitted to obtain the impulse response coefficients of an aircraft system considered as a whole for a particular type of operation. Using this model, the output which is the aircraft response can be obtained with lesser computation time for any runway profile as the input.

I. Introduction

DETERMINATION of the dynamic response of an aircraft to runway unevenness during takeoff, landing, and taxiing can be done either in time or frequency domain. In the time domain analysis,[1,2] the dynamic equations representing the aircraft characteristics are formed, and knowing the operational qualities and the elevation profile of the runway, the response is computed. In the time domain analysis, system operational quantities such as nonlinearities of the shock absorber system characteristics, control surface effects, and variation of lift with forward velocity of the aircraft can be incorporated in the mathematical model without difficulty. Dynamic increments due to discrete runway bumps can be evaluated conveniently using this approach. The only disadvantage in applying this technique is that it consumes considerable computer time for the computation of the response.

A statistical study of stationary unevenness effects on the response of an aircraft can be helpful to the design engineer, particularly in evaluating the fatigue damage of the structure. This study is made simple using the power spectral approach or frequency domain approach. Knowing the transfer function of the aircraft and the power spectrum of runway elevation profile, the response statistics can be computed. The potentiality of this approach can be extended to nonlinear systems by applying equivalent linearization and perturbation methods. Various authors[1,3] have applied this technique successfully for a simple two or three degrees-of-freedom aircraft model which is traveling with a steady forward velocity. If more degrees of freedom representing the aircraft flexibilities are to be considered, the problem becomes complex. Extra care has to be taken to analyze systems when the input is nonstationary, and the input becomes nonstationary when the aircraft is accelerating or decelerating. The time series approach, even though belonging to the time domain technique, eliminates to a considerable extent the disadvantages of the usual techniques. It basically consists of fitting a simple mathematical model by relating the input (runway elevation profile) and the output of interest (for example, pilot location acceleration, acceleration at the center of gravity, loads at certain critical points, etc.). This simple mathematical model represents the impulse response of the system (aircraft) considered as a whole. Here the modeling of the various subsystems do not come into the picture, thus eliminating the problems involved in tackling the individual nonlinearities. To such a model various runway profiles can be fed in as inputs and the output will give the response of the aircraft as though it were operating on different runways. An idea about the runway roughness can then be had. The case studied in this paper is the aircraft (Boeing 733-94) taxiing with a steady forward velocity of 100 ft/s.[1]

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II. Preliminaries

The autocovariance function (acf) of a discrete stationary stochastic process x_i at lag h is defined by

$$C_x(h) = E(x_i - \mu_x)(x_{i+h} - \mu_x) \qquad h = 0,1,\ldots,N-1 \tag{1}$$

where

$$E(x_i) = E(x_{i+h}) = \mu_x$$

At lag zero, the acf becomes the variance, σ_x^2 of the process. The normalized autocovariance function (nacf) is defined by

$$r_x(h) = C_x(h)/C_x(0) \qquad h = 0,1,\ldots,N-1 \tag{2}$$

Under the assumption of ergodicity (ensemble average = time average), the acf and nacf can be estimated[4] from a sample realization of the ensemble. The estimated acf, $\hat{C}_x(h)$, at lag h is expressed by

$$\hat{C}_x(h) = \frac{1}{n-h} \sum_{i=1}^{N-h} (x_i - \hat{\mu}_x)(x_{i+h} - \hat{\mu}_x)$$

$$h = 0,1,\ldots,N-1 \tag{3}$$

where

$$\hat{\mu}_x = \frac{1}{N} \sum_{i=1}^{N} x_i \tag{4}$$

and N is the number of samples.

For large N, $\hat{C}_x(h)$ is an unbiased estimator for acf $C_x(h)$. Similarly, the unbiased estimator for nacf is

$$\hat{r}_x(h) = \hat{C}_x(h)/\hat{C}_x(0) \qquad h = 0,1,\ldots,N-1 \tag{5}$$

Similar relationships can be defined for the cross-covariance functions of two random processes.[4,5]

If the nacf is zero beyond lag q, then the variance of the estimated nacf, $\hat{r}_x(h)$, beyond lag q is given by Bartlett's relationship.[5,6]

$$\operatorname{Var}\hat{r}_x(h)=\frac{1}{N-h}\left[1+2\sum_{g=1}^{q}r_x^2(g)\right]\quad |h|>q \tag{6}$$

More general expressions for the variance can be found in Ref. 7.

The standard error (SE) of the estimated nacf is the square root of the variance

$$\text{SE of } \hat{r}_x(h)=\left[\frac{1}{N-h}\left[1+2\sum_{g=1}^{q}r_x^2(g)\right]\right]^{1/2}\quad |h|>q \tag{7}$$

Since the nacf is not known, the standard error can be determined by substituting the estimated nacf. Once the standard error for a particular lag h is known, and if the estimated nacf for lag h is less than this value, then that nacf at lag h can be concluded to be zero within certain confidence limits.

III. Modeling

Depending on the type of time series record available, modeling can be accomplished by autoregressive (AR), moving average (MA), or autoregressive moving average (ARMA) processes under the assumption of stationarity. These processes are briefly explained below with reference to Fig. 1.

In Fig. 1 the system is assumed to be either a linear or an equivalent linear system of the original system.

The input v_i to the system is assumed to be a white noise sequence. The AR process of order p is expressed by

$$Y_i=\sum_{k=1}^{p}\phi_k B^k Y_i+v_i\qquad i=1,2,\ldots,N-1 \tag{8}$$

where B^k is the backward shift operator defined by

$$B^k Y_i=Y_{i-k}$$

and ϕ_k are the weighting coefficients.

Preliminary estimates of the weighting coefficients can be obtained by solving the Yule-Walker equations[4]

$$\begin{Bmatrix} C_{yy}(1)\\ C_{yy}(2)\\ \vdots\\ C_{yy}(p)\end{Bmatrix}=\begin{bmatrix} C_{yy}(0) & C_{yy}(1) & \cdots C_{yy}(p-1)\\ C_{yy}(1) & C_{yy}(0) & \cdots C_{yy}(p-2)\\ \vdots & \vdots & \vdots\\ C_{yy}(p-1) & C_{yy}(p-2) & \cdots C_{yy}(0)\end{bmatrix}\begin{Bmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_p\end{Bmatrix} \tag{9}$$

which are linear in ϕ_k.

The preliminary estimates of the ϕ_k's are used as starting values in the maximum likelihood estimation algorithm[4,8] to obtain maximum likelihood estimates of ϕ_k.

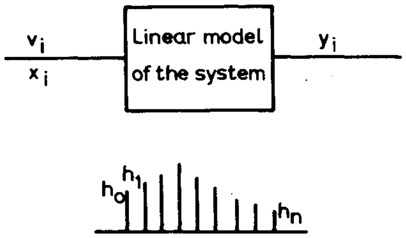

Fig. 1 System and impulse response sequence.

The moving average model of order q is given by

$$Y_i=v_i-\sum_{k=1}^{q}\theta_k B^k v_i\qquad i=1,\ldots,N-1 \tag{10}$$

and the mixed ARMA (p,q) process is given by

$$Y_i-\sum_{k=1}^{p}\phi_k B^k Y_i=v_i-\sum_{k=1}^{q}\theta_k B^k v_i\qquad i=1,\ldots,N-1 \tag{11}$$

In both of these processes preliminary estimates of ϕ_k and θ_k can be obtained by search techniques only. A maximum likelihood estimation algorithm to obtain ϕ_k and θ_k is given in Ref. 4.

If the process is nonstationary, the data can be differenced until stationarity results. If Δ is the backward difference operator, the differenced data ΔY_i are obtained from $\Delta Y_i=Y_i-Y_{i-1}$. A reasonable indication of nonstationarity is the variance. If the variance decreases with differencing, then the process is concluded to be nonstationary. If a stationary process is differenced, then the variance increases. Thus, any given process is differenced up to the point where the variance just starts to increase. The resulting stationary process is then used for modeling. If the input is different from white noise as in the present case of an aircraft response, a discrete linear transfer function model can be identified.

X_i and Y_i are input and output signals discretized at equal intervals (Fig. 1). From the discrete signals it is possible to obtain the impulse response sequence of the dynamic system. In the present study, X_i represents the runway elevation about a datum and Y_i represents the pilot location vertical acceleration. The input and output sequences are related by

$$Y_i=\sum_{k=0}^{\infty}h_k B^k X_i$$

$$=h(B)X_i \tag{12}$$

The weights h_0, h_1,... are the impulse response coefficients. In practice, the system will be affected by disturbances or noise, whose net effect is to corrupt the output predicted by the transfer function model by an amount N_i. The complete transfer function model can be postulated as

$$Y_i=h(B)X_i+N_i \tag{13}$$

where the noise N_i is not necessarily white.

Since the impulse response function does not vary with differencing, let

$$y_i=Y_i-Y_{i-1}=\Delta Y_i$$

$$x_i=X_i-X_{i-1}=\Delta X_i$$

and differencing Eq. (12) yields

$$y_i=h(B)x_i \tag{14}$$

For the system to be stable, the series must converge under the condition $|B|\leq 1$. The stability condition implies that a finite incremental change in the input results in a finite incremental change in the output. Now, suppose X is held constant indefinitely, then the output must reach a steady state such that

$$Y_\infty=GX \tag{15}$$

where G is defined as the steady-state gain. From Eq. (12), if $X_i=+1$ for all t,

$$Y_\infty=(h_0+h_1+h_2+\cdots)1 \tag{16}$$

and from Eqs. (15) and (16),

$$G=\sum_{i=0}^{\infty} h_i \tag{17}$$

In the present case, the input X is a displacement and if X is constant then Y, the output which is acceleration, has to be necessarily zero. As a consequence, $G=0$. This condition is an important one in the sense that it gives the point where the impulse response sequence can be truncated.

IV. Identification of Impulse Response by Prewhitening

Impulse response coefficients can be determined by prewhitening the input. Suppose a suitably differenced input process x_i is stationary, it is possible to represent the series x_i by an ARMA process

$$\phi_x(B)\theta_x^{-1}(B)x_i=v_i \tag{18a}$$

$$\phi_x(B)=1-\sum_{k=1}^{p}\phi_k B^k \tag{18b}$$

$$\theta_x(B)=1-\sum_{k=1}^{q}\theta_k B^k \tag{18c}$$

The preceding relation can be thought of as transforming the correlated signal x_i into an uncorrelated noise v_i. If the same number of differencing performed on x_i to get stationary x_i is performed on Y_i and N_i, Equation (13) becomes

$$y_i=h(B)x_i+n_i \tag{19}$$

Multiplying both sides of Eq. 19 by $\phi_x(B)\theta_x^{-1}(B)$ and substituting Eqs. (18) results in

$$\phi_x(B)\theta_x^{-1}(B)y_i=h(B)v_i+\phi_x(B)\theta_x^{-1}(B)n_i \tag{20}$$

By defining $z_i=\phi_x(B)\theta_x^{-1}(B)y_i$ and $\epsilon_i=\phi_x(B)\theta_x^{-1}(B)n_i$, Eq. (20) can be written as

$$z_i=h(B)v_i+\epsilon_i \tag{21}$$

The cross covariance between v_i and z_i can be obtained by multiplying both sides of Eq. (21) by v_{i-k} and taking expectations

$$C_{vz}(k)=h_k\sigma_v^2$$

or

$$h_k=\frac{C_{vz}(k)}{\sigma_v^2} \qquad k=0,1,\ldots \tag{22}$$

Hence from Eq. (22) it is possible to get the values of the impulse response coefficients h_k for any value of k.

The following is the procedure adopted for the calculation of the impulse response coefficients. First, the runway profile is fitted with an AR(1) model per Eq. (8). Using this model, the pilot location acceleration is modified to z_i and using Eq. (22), the impulse response coefficients are estimated. The impulse response function is terminated at N, where N, the number of terms, is obtained from

$$\sum_{i=0}^{N} h_i=G=0 \tag{23}$$

Two sets of the impulse response coefficients of the aircraft are computed from the responses of the aircraft operating identically on two different runways. It is shown that for a given runway profile input, both the impulse response coefficient sets give statistically the same output. It can be concluded that both the impulse response coefficient sets are statistically the same.

V. Results and Discussion

The procedure developed above is used to obtain the impulse response of an aircraft system taken as a whole when the aircraft is traversing a runway. The input to the aircraft is the runway elevation profile which is represented as follows. Knowing the distance s along the runway, described by

$$s=ut+1/2at^2 \tag{24}$$

where u is the initial velocity, t the time, and a the acceleration of the aircraft, the elevation profile corresponding to a certain s can be obtained from the elevation profile data. These profile data, which are discretized at equal intervals of time, are taken as the input to the aircraft model. The output can be any response of interest. As an example, in the present study an aircraft taxiing a runway with a steady forward velocity is considered. The output signal is the vertical acceleration response at the pilot location. The aircraft considered for this analysis is Boeing 733-94.[1] The input X is the runway elevation given in Figs. 2 and 3. The pilot location acceleration response for this aircraft traversing over runways 1 and 2 under a steady velocity of 100 ft/s is given in Figs. 4 and 5. Symbols R1 and R2 refer to various quantities related to runways 1 and 2, respectively. The signals are discretized at an interval of 0.05 s. Table 1 presents the statistics of pilot location acceleration response of the aircraft using the complete dynamics model when it is traversing runways 1 and 2 under a steady velocity. The mean and variance in both cases can be seen to be almost of the same level. The same conclusion can be arrived at by using statistical tests for determining the homogeneity of the means and variances. Hence, both runways can be classified to have the same effect on the aircraft.

Figures 6 and 7 give the nacf of the acceleration response for the two runways. It can be seen that there is a periodicity in the nacf and it is roughly of the value 9. These nine lags correspond to a time of 0.45 s and the frequency is 2.22,

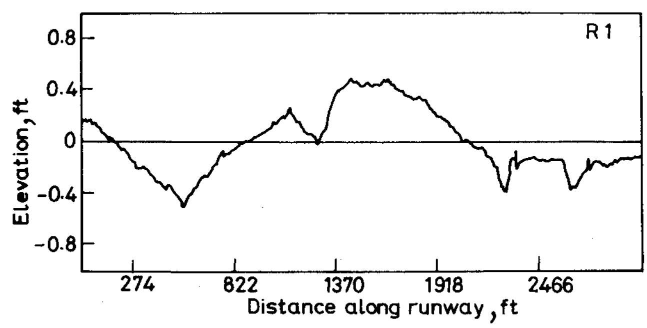

Fig. 2 Runway elevation profile.

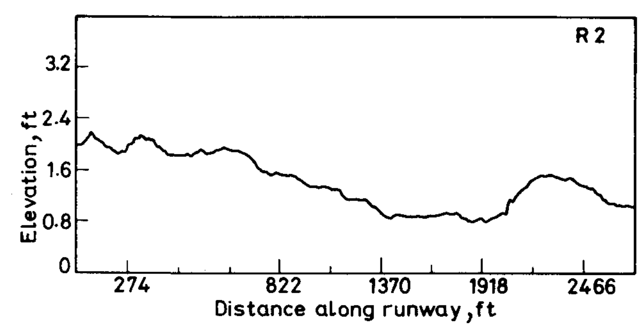

Fig. 3 Runway elevation profile.

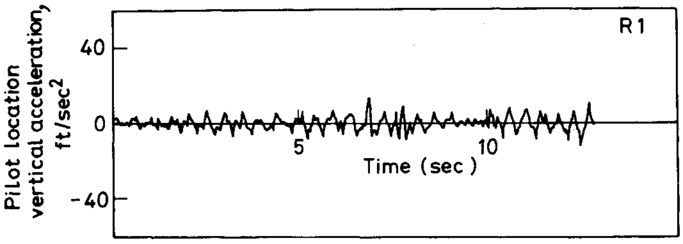

Fig. 4 Time response of vertical acceleration at pilot location.

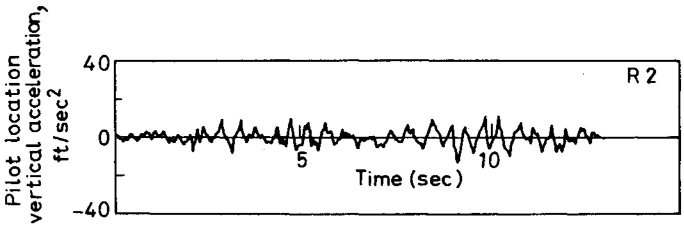

Fig. 5 Time response of vertical acceleration at pilot location.

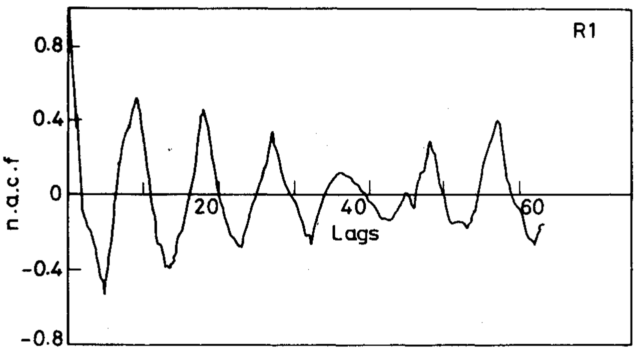

Fig. 6 Normalized autocovariance function of acceleration response.

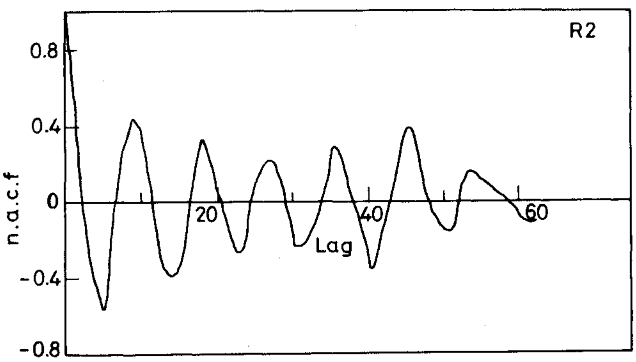

Fig. 7 Normalized autocovariance function of acceleration response.

Table 1 Results of dynamic analysis

	Vertical acceleration response at pilot location		
Input	Mean, ft/s^2	Variance, (ft/s^2)^2	Standard deviation ft/s^2
Runway 1	−0.121	15.329	3.915
Runway 2	−0.015	16.181	4.022

Table 2 Runway models, $\Delta R_i = \phi \Delta R_{i-1} + v_i$

Input	ϕ	$\sigma_v^2, \times 10^{-3}$
Runway 1	0.229	0.21
Runway 2	0.529	0.12

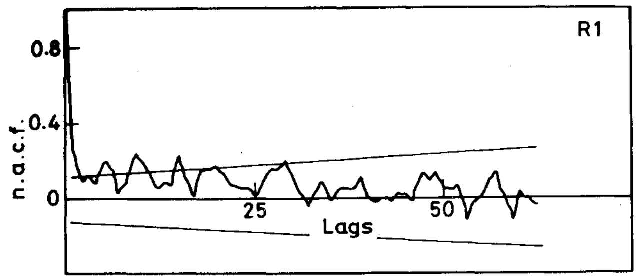

Fig. 8 Normalized autocovariance function of differenced data of runway profile.

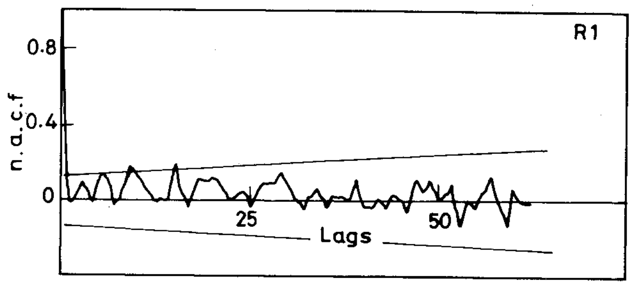

Fig. 9 Normalized autocovariance function of residuals after modeling runway profile.

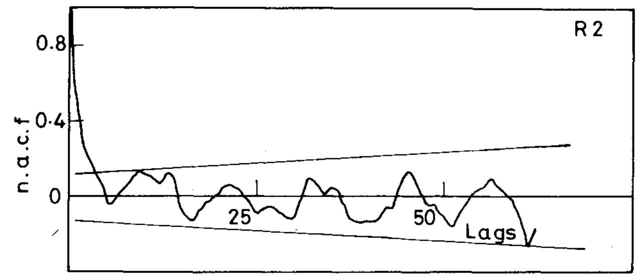

Fig. 10 Normalized autocovariance function of differenced data of runway profile.

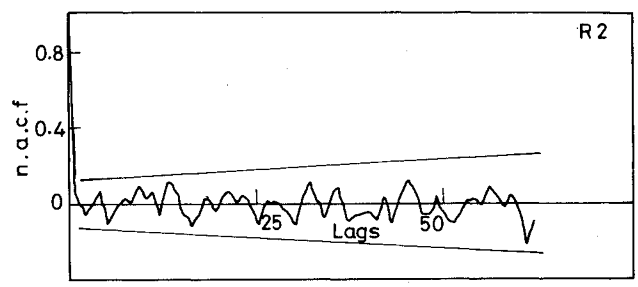

Fig. 11 Normalized autocovariance function of residuals after modeling runway profile.

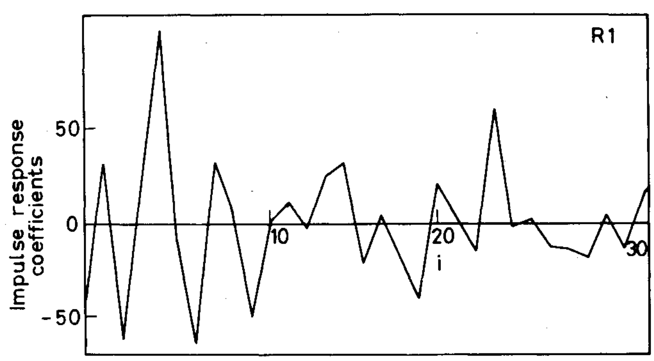

Fig. 12 Impulse response coefficients.

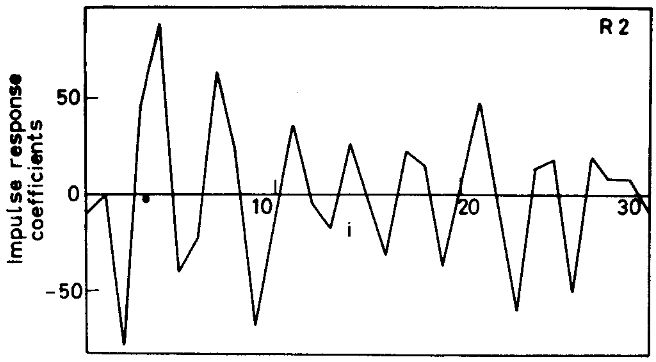

Fig. 13 Impulse response coefficients.

Table 3 Impulse response coefficients

	h_i	
i	Runway 1 $(h_{R1})_i$	Runway 2 $(h_{R2})_i$
0	−39.21	−10.81
1	32.98	−1.262
2	−63.64	−82.77
3	14.06	47.22
4	102.7	93.01
5	−11.47	−42.22
6	−62.72	−26.03
7	32.02	68.01
8	6.297	13.61
9	−49.15	−7.229
10	−1.153	−25.47
11	7.72	39.87
12	−3.07	2.47
13	26.38	−23.46
14	31.44	27.98
15	−21.9	−8.139
16	3.786	−34.13
17	−21.19	26.48
18	−40.34	16.62
19	21.78	−42.53
20	1.554	9.659
21	−15.87	54.26
22	60.83	−2.159
23	−1.175	−66.61
24	0.777	16.27
25	−13.27	22.01
	0.472	−0.411

Table 4 Results obtained using time series model

		Acceleration response at pilot location			
Input	Impulse response coefficients	Mean, μ	Variance, σ^2	Standard deviation, σ	Standard deviation of mean, $\sigma_{\hat{\mu}}$
Runway 1	$(h_{R1})_i$	0.070	9.7	3.11	0.19
Runway 2	$(h_{R1})_i$	−1.27	7.05	2.66	0.17
Runway 1	$(h_{R2})_i$	0.24	5.58	2.36	0.15
Runway 2	$(h_{R2})_i$	−1.08	5.61	2.37	0.15

which is found to be almost equal to the predominant natural frequency 2.196 for the system.

The runway profile data R_i have to be modeled first in order to calculate the impulse response coefficients. The data R_i are differenced once to induce stationarity, i.e., $\Delta R_i = R_i - R_{i-1}$. The differenced data ΔR_i is fitted with a first-order AR(1) process.

$$\Delta R_i = \phi_R \Delta R_{i-1} + v_i \tag{25}$$

The values of ϕ_R for runways 1 and 2 are 0.229 and 0.529, respectively (Table 2). Since both runways are fitted with AR(1) model, they can be classified as similar types of runways. The variances σ_v^2 of the residuals v_i for both runways are 0.21×10^{-3} and 0.12×10^{-3}. Figures 8-11, present the nacf of the differenced runway profile data and the residual noise v_i along with the standard error limit obtained from Bartlett's formula, Eq. (6), with $q = 0$.

The impulse response coefficients are calculated for the aircraft from the acceleration response obtained in both runways. Using the constraint that

$$\sum_{i=0}^{N} h_i = 0$$

the impulse response function is terminated at 26 terms (i.e., $N = 25$). The sum of the first 26 terms in both cases gives a value of 0.47 and −0.41. The impulse response coefficients are given in Table 3 and Figs. 12 and 13.

Using the transfer function model for runways 1 and 2, the response is calculated with both runways as the inputs, and the statistics of the response are presented in Table 4. With runway 1 profile as the input and using the impulse response coefficients determined for runway 1, the mean and variance are 0.07 and 9.7, respectively. Using the same runway 1 profile as input, but using the impulse response coefficients determined for runway 2, the mean and variance are 0.24 and 5.58, respectively. In a similar manner, with runway 2 profile as the input and using the impulse response coefficients determined for runway 1, the mean and variance are −1.27 and 7.05, respectively; using the impulse response coefficients determined from runway 2, the mean and variance are −1.08 and 5.61.

To insure that these two sets obtained above are statistically the same, the variance of the estimated mean $\sigma_{\hat{\mu}}^2$ of the process is calculated using the relation

$$\sigma_{\hat{\mu}}^2 = \frac{\sigma^2}{N}$$

where $\sigma_{\hat{\mu}}^2$ is the variance of the estimated mean, σ^2 the variance of the output process obtained by using impulse response coefficients, and N the total number of points (256).

For case 1, where the input is runway 1 profile, $\sigma_{\hat{\mu}}$ is 0.19 (i.e., $3.11/\sqrt{256}$). It can be seen that the means obtained for the two impulse response coefficient sets, namely 0.07 and 0.24, are within $\pm 2\sigma_{\hat{\mu}}$ (i.e., ±0.38) limits providing 95% confidence limit.

Similarly for case 2, where the input to the models is runway profile 2, $\sigma_{\hat{\mu}}$ (i.e., $2.66/\sqrt{256}$) is 0.17 and the means are −1.27 and −1.08. Here also, both of these mean values fall within $\pm 2\sigma_{\hat{\mu}}$ limits.

The above observations lead to a conclusion that the impulse response model of the aircraft obtained from both runways are statistically the same. Since both the runway models are AR(1) and the residual variance $\hat{\sigma}_v^2$ is also of the same order, these two are assumed to be two samples of an ensemble. As a consequence, the response of the aircraft is also of the same order for both runways. (This is confirmed by the results presented in Tables 1 and 4.)

In the time domain analysis, computation of acceleration response of the aircraft (Figs. 4 and 5), traversing a runway length of 1250 ft with a steady forward velocity of 100 ft/s, using the dynamic equations, consumes 6 min of computer time (IBM 360/44). But this computation time is reduced to 1 min by using the impulse response model of the aircraft obtained from the time series analysis. The difference between the variance of the calculated response using the transfer function model and the actual dynamics (Table 1) is attributed to the noise N_i introduced at the output level [Eq. (13)]. The present study is only an attempt to illustrate the broad justification in the application of time series techniques to aircraft taxiing problems and therefore modeling of noise is not considered in this investigation.

VI. Conclusion

Time series analysis is a technique that can be used to model an aircraft traversing a runway. This simple mathematical model represents the impulse response of the aircraft system as a whole, thus eliminating the problems involved in handling the individual nonlinearities of the system. Since this model relates a single output to a single input, one requires different impulse response sequences corresponding to various outputs in a multioutput system. For a given aircraft and a runway, an output signal of interest can be computed, with less computation time, using the relevant impulse response sequence.

In the opinion of the authors, the coefficient obtained while modeling the runway [Eq. (25)] must have a correlation to the power spectral density of the runway as obtained by Houbolt,[9] using prewhitening and postdarkening techniques. Therefore, one may be led to the classification of runways based on these models.

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